ZIBELINE INTERNATIONAL PUBLISHING
SSN: 2521-0904 (Print)
ISSN: 2521-0440 (Online)
CODEN: EHJNA9

Engineering Heritage Journal (GWK)

DOI: http://doi.org/10.26480/gwk.01.2020.01.03

RESEARCH ARTICLE

POLYNOMIAL SERIES SOLUTION OF BENJAMIN–BONA–MAHONY EQUATION VIA DIFFERENTIAL TRANSFORM METHOD

Muhammad Jamil*, Badshah-E-Room

Department of Mathematics, University of Malakand, Chakdara Dir(L), KPK, Pakistan.
****Corresponding Author Email: jamilkanju@gmail.com***

ARTICLE DETAILS

Article History:

Received 25 February 2020
Accepted 28 March 2020
Available online 21 April 2020

ABSTRACT

In this work, we solved the initial value problem of Benjamin–Bona–Mahony equation with generalized initial conditions by using differential transform method (DTM). We obtained the general solution in the form of a series. An example is presented for the particular initial conditions.

KEYWORDS

Differential Transform Method, Initial Value Problem, Benjamin–Bona–Mahony Equation.

1. INTRODUCTION

Non-linear differential equations occurs in many areas of applied mathematics, engineering, physics and other sciences. Non-linear phenomena is in the interest of applied mathematicians, physicist and engineers. Most of the mathematical systems are inherently having non-linear nature. Due to numerous applications, NLODEs have received much attention. As dynamical non-linear ODEs are difficult to solve, commonly non-linear systems are approximated to linear systems, i.e. linearization takes place for solutions of non-linear systems. Generally linear and non-linear phenomenon occur in most of the recent fields of modern sciences and engineering. In such areas most of the problems are modeled by partial differential equations (PDEs). So determining solutions of such partial differential equations has an important role in science problems.

A great deal of research has been made to handle PDEs and many methods has been established to obtain the exact solution such as tanh-coth method, the tanh-sech method, sine-cosine method, the exp-function method, the sn-ns method (Wazwaz, 2007; Wazwaz, 2006; Yadong, 2005; He and Zhang, 2008; Alvaro, 2012). Different authors used different approaches to solve different linear and non-linear differential equations. In a study, the author used laplace transform method to solve fractional order heat equation (Khan et al., 2017). Eltayab used adomian decomposition method for the solution of non-linear partial differential equations (Eltayab, 2017).

In this paper, we use Differential transform method to solve Benjamin-Bona-Mahony equation (Muatjetjeja et al., 2014). The Benjamin-Bona-Mahony equation is a non-linear partial differential equation which is given by

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

This equation was firstly introduced in 1972 as an alternative to the well-known Korteweg de Vries (KdV) equation given by (Benjamin et al., 1972):

$$u_t + u_x + u_{xxx} + uu_x = 0.$$

The basic idea of differential transform (DTM) was initially introduced (Ertrk, 2007; Zhou, 1986). The main application of this technique was to solve both linear and nonlinear initial value problems in electrical circuit's analysis. In this method, transformation rules are applied to the differential equations and the initial or boundary conditions, the system are transformed into a set of algebraic equations in terms of the differential transforms of the original functions and the solution of these algebraic equations gives the desired solution of the problem. It gives exact values of the kth derivative of an analytical function at a point in terms of known and unknown boundary conditions in a fast manner. It is a technique for the solutions of the differential equations in the form of polynomial series.

It is different from the higher order Taylor series method which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method is computationally expensive, especially for high order equations. The differential transform is an iterative procedure for obtaining analytic series solution. In recent years, the researchers have applied the method to various linear and nonlinear problems such as integro-differential equations, partial differential equations, two-point boundary value problems, KdV and mKdV equations, we refer the readers to (Ertrk, 2007; Zhou, 1986; Jang et al., 2001; Alquran, 2012) and the references therein. In next two sections we describe the method of differential transformation. In section 2 we give differential transform method (DTM) for one dimensional time dependent functions and in section 3 we present DTM for two variables functions.

Quick Response Code	Access this article online	
	Website: www.enggheritage.com	**DOI:** 10.26480/gwk.01.2020.01.03

Cite the Article: Muhammad Jamil, Badshah-E-Room (2020). Polynomial Series Solution Of Benjamin–Bona–Mahony Equation Via Differential Transform Method. *Engineering Heritage Journal, 4(1): 01-03.*

2. Differential Transform Method For a Time Variable Function

Differential transformation of the k^{th} derivative of time (t) dimensional function $u(t)$ is given by:

$$U(k) = \frac{1}{k!}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0} \quad (1)$$

where $u(t)$ is original function and $U(k)$ is the transformed function.

Inverse differential transform of $U(k)$ is defined as:

$$u(t) = \sum_{k=0}^{\infty} U(k)(t-t_0)^k. \quad (2)$$

Combining (1) and (2), we have

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{d^k u(t)}{dt^k}\right]_{t=t_0}\frac{(t-t_0)^k}{k!}. \quad (3)$$

If $t_0 = 0$, then the function $u(t)$ in equation (3) can be expressed as

$$u(t) = \sum_{k=0}^{\infty}\left[\frac{d^k u(t)}{dt^k}\right]_{t=0}\frac{t^k}{k!}. \quad (4)$$

As a result of the above definition various differential transforms of time dimensional functions are listed in a table below:

Table 1: Various time dimensional differential transformations.

Original function	Transform function
$w(t) = u(t) + v(t)$	$W(k) = U(k) + V(k)$
$u(t) = cv(t)$	$U(k) = cV(k)$
$u(t) = \frac{d^n v(t)}{dx^n}$	$U(k) = \frac{(k+n)!}{k!} V(k+n)$
$w(t) = u(t)v(t)$	$W(k) = \sum_{l=0}^{k} U(l)V(k-l)$
$u(t) = t^n$	$U(k) = \delta(k-n) = \begin{cases} 0 & if\ k \neq n \\ 1 & if\ k = n \end{cases}$
$U(t) = \int_0^t v(\eta)\, d\eta$	$U(k) = \frac{V(k-l)}{k}, where\ k \geq 1$
$x(t) = u(t)v(t)w(t)$	$X(k) = \sum_{s=0}^{k}\sum_{m=0}^{k-s} U(s)V(m)W(k-s-m)$
$u(t) = e^{at}$	$U(k) = \frac{a^k}{k!}$
$u(t) = \sin at$	$U(k) = \frac{a^k}{k!}\sin\frac{k\pi}{2}$
$u(t) = \cos at$	$U(k) = \frac{a^k}{k!}\cos\frac{k\pi}{2}$

3. Differential Transform Method For Two Variable Function

Differential transformation of the $(k+h)^{th}$ partial derivative $\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t}$ of a function $u(x,t)$ is defined as

$$U(k,h) = \frac{1}{k!h!}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t}\right]_{(x_0,t_0)}, \quad (5)$$

where $u(x,t)$ is original function and $U(k,h)$ is the transformed function. Inverse differential transform of $U(k,h)$ is defined as

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)(x-x_0)^k (t-t_0)^h. \quad (6)$$

Combining (5) and (6), we have

$$u(t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t}\right]_{(x_0,t_0)}\frac{(x-x_0)^k(t-t_0)^h}{k!h!}. \quad (7)$$

If $(x_0,t_0) = (0,0)$, then the function $u(x,t)$ in equation (7) can be expressed as

$$u(t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\left[\frac{\partial^{k+h}u(x,t)}{\partial x^k \partial t}\right]_{(0,0)}\frac{x^k t^h}{k!h!}. \quad (8)$$

As a result of the above definition various differential transforms of two variable functions are listed in a table below:

Table 2: Various differential transformations of two variables functions.

Original function	Transform function
$w(x,t) = u(x,t) + v(x,t)$	$W(k,h) = U(k,h) + V(k,h)$
$u(x,t) = cv(x,t)$	$U(k,h) = cV(k,h)$, c is a constant
$u(x,t) = \frac{\partial^n v(x,t)}{\partial x^n}$	$U(k,h) = \frac{(k+n)!}{k!} V(k+n,h)$
$u(x,t) = \frac{\partial^n v(x,t)}{\partial t^n}$	$U(k,h) = \frac{(h+n)!}{h!} V(k,h+n)$
$w(x,t) = u(x,t)v(x,t)$	$W(k,h) = \sum_{i=0}^{k}\sum_{j=0}^{h} U(i,h-j)V(k-i,j)$
$u(x,t) = x^m t^n$	$U(k) = \delta(k-m,h-n) = \begin{cases} 1 & if\ k=m, h=n \\ 0 & otherwise \end{cases}$
$u(x,t) = \frac{\partial^{r+s} v(x,t)}{\partial x^r \partial t^s}$	$U(k,h) = \frac{(k+r)!(h+s)!}{k!h!} V(k+r,h+s)$

4. Main Result

We consider the Benjamin-Bona-Mahony equation

$$u_t + u_x + uu_x - u_{xxt} = 0, \quad (9)$$

with initial conditions

$$u(x,0) = f(x) \text{ and } u_t(x,0) = g(x). \quad (10)$$

Taking Differential Transform of equation (9), we have

$$\begin{cases} (h+1)U(k,h+1) + (k+1)U(k+1,h) \\ +\sum_{i=0}^{k}\sum_{j=0}^{h}(k-i+1)U(i,h-j)U(k-i+1,j) \\ -(k+1)(k+2)(h+1)U(k+2,h+1) = 0. \end{cases} \quad (11)$$

Now taking differential transform of the initial conditions (10), we have

$$U(k,0) = F[k], \quad (12)$$

$$U(k,1) = G[k], \quad (13)$$

where $F[k]$ and $G[k]$ are one dimensional differential transforms of functions $f(x)$ and $g(x)$ respectively. We can evaluate $U(k,h)$ for all non-negative integers k and h, by using equations (11) - (13). Then by equation (6), the desired polynomial series solution at $(x_0,t_0) = (0,0)$ will be in the form

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} U(k,h)x^k t^h. \quad (14)$$

5. Examples

In this section we give an example which will illustrate the main result.

Example 5.1. Consider that in initial conditions (10), $f(x) = e^x$ and $g(x) = \sin x$, then equations (12) and (13) gives:

$$U(k,0) = \frac{1}{k!}, \tag{15}$$

$$U(k,1) = \frac{1}{k!}\sin\frac{k\pi}{2}, \tag{16}$$

Using (11) and (14)-(16), we have

$$u(x,t) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots\right) + xt + \frac{xt^2}{2} + \frac{xt^3}{3} + \cdots,$$

which is the required series solution of Benjamin-Bona-Mahony equation.

6. Conclusions

In this study, we concerned mainly with application of differential transformation method (DTM) for initial value problem of non-linear partial differential equation, named Benjamin-Bona-Mahony equation. The study showed that by using DTM one can easily obtain solution for the PDEs problems described above. We obtained the solution in a form of two variables polynomial series. An example is provided as illustration.

References

Alquran, M.T., 2012. Applying differential transform method to nonlinear partial differential equations: A modified approach. Applic. Applied Math. Int. J., 7, Pp. 155-163.

Alvaro, H.S., 2012. Solving nonlinear partial differential equations by the sn-ns method. Abstr. Appl. Analysis, 25, Pp. 1-25.

Benjamin, T.B., Bona, J.L., Mahony, J.J., 1972. Model equations for long waves in nonlinear dispersive systems. Philos. Trans. R. Soc. Lond. A, 272, Pp. 4778.

Eltayab, H., 2017. A note on double Laplace decomposition method and nonlinear partial differential equations. New Trends in Mathematical Sciences, 4, Pp. 156-164.

Ertrk, V.S., 2007. Differential Transformation Method for solving differential equations of Lane-Emden type. Mathematical and Computational Applications, 12 (3), Pp. 135-139.

He, J.H., Zhang, L.M., 2008. Generalized solitary solution and compacton-like solution of the Jaulent-Miodek equations using the Exp-function method. Physics Letters A, 372, Pp. 1044-1047.

Jang, M.J., Chen, C.L., Liu, Y.C., 2001. Two-dimensional differential transform for partial differential equations. Applied Math. Comput., 121, Pp. 261-270.

Khan, T., Shah, K., Khan, A., Khan, R.A., 2017. Solution of fractional order heat equation via triple Laplace transform in two dimensions. Mathematical Methods in the Applied Sciences, 41 (2), Pp. 1099-1476.

Muatjetjeja, B., Khalique, C.M., Mahony, B.B., 2014. Equation with Variable Coefficients: Conser- vation Laws. Symmetry, 6, Pp. 1026-1036.

Wazwaz, A.M., 2006. New travelling wave solutions of different physical structures to generalized BBM equation. Physics Letters A, 355, Pp. 358-362.

Wazwaz, A.M., 2007. The extended tanh method for new solitons solutions for many forms of the fifth order KdV equations. Appl. Math. Comput., 184, Pp. 1002- 1014.

Yadong, J.S., 2005. Explicit and exact special solutions for BBM-like B(m;n) equations with fully nonlinear dispersion. Chaos, Solitons and Fractals, 25, Pp. 1083-1091.

Zhou, J.K., 1986. Differential Transformation and its Applications for Electrical Circuits. Huazhong University Press, Wuhan, China.