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RESEARCH ARTICLE

A STUDY OF PSEUDO SPECTRAL GALERKIN METHOD FOR SOLVING DIFFERENTIAL EQUATIONS

Qura tul Ain[b], Muhammad Ali[a], Bilqees[a], Muhammad Yousif[a]

[a]National Engineering Laboratory for Modern Silk, College of Textile and Engineering, Soochow University, Suzhou, P.R. China
[b]School of Mathematical Sciences, Soochow University, Suzhou, P.R. China
***Corresponding Author e-mail:** mzaqau@hotmail.com*

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ABSTRACT

Differential equations have a remarkable potential to exhibit real life phenomenon. Many methods have been developed to solve these differential equations, though only a few stands with time. This paper presents a comparison of Pseudo Spectral Galerkin Method for solving ordinary differential equations with many other global methods. Results shows the high accuracy and rapid convergence of said method. Graphical comparison and error tables have been provided for better understanding of results.

KEYWORDS

Pseudo Spectral Galerkin Method, equations, rapid convergence

1. INTRODUCTION

Differential equations provide a surprising predictability of the world around us. They are used from textile, physics, hydrology and engineering in a broad range of disciplines (ul Ain et al., 2019; Ullah et al., 2019; Rehman et al., 2019; Ali et al., 2019). They are of utmost importance in the present days and have revolutionary contribution in the progress of modern age. But with knowing the history of differential equations, we cannot proceed further.

In the view of some mathematician, Gottfried Wilhellm von Leibniz (1646-1716) laid the stone of differential equations in 1675. He wrote an integral equation $\int x\,dx = \frac{1}{2}x^2$ (Keller, 1976). While Issac Newton (1642-1742) classified differential equations into three classes as under (Ince, 1956).

(i) $\frac{dy}{dx} = f(x)$
(ii) $\frac{dy}{dx} = f(x, y)$
(iii) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial x} = f$

We can see that the equations (i) and (ii) are ordinary differential equations of one and two dependent variables, whereas the third one is partial differential equation involving more one independent variable.

Later on, Bernoulli's family (1623-1789) work spanned over late seventeenth and eighteenth century; method for solving first order differential equations took place in Bernoulli's age, while differential equations of second and third order dealt in the beginning of eighteenth century (John; Bittanti and Plitecnico; Bernoulli, 1742). John Bernoulli discusses the general solution of the equation for parabolic and hyperbolic form. Jacopo Riccati's (1676-1754) introduced two new differential equations, written in present symbols are the early form of Riccati's differential equation.

$\frac{d^2y}{dx^2} = \frac{2y}{x}$,
$\dot{x} = \alpha x^2 + \beta t^m$
$\dot{x} = \alpha x^2 + \beta t + \gamma t^2$

The problem of reducing a particular class of second order differential equation to first order was treated by Leonhard Euler (1707-1783). He also treated homogeneous linear differential equation with constant coefficients. Euler uses the method of integrating to solve the differential equations reduced from second order to first order one. Joseph-Louis Lagrange (1736–1813) showed that the general solution of nth order linear homogeneous differential equations is the linear combination of linearly independent solutions. He is also known for the fundamental work of partial differential equation and calculus of variation.

By the end of eighteenth century, elementary methods for the solution of ordinary differential equations were discovered. The brief work of Euler, Ritz, and Galerkin can be seen in Martin J. Gander., Gerhard Wanner (Martin and Gerhard, 2012). In that paper Martin and Wanner gives a complete detail about the construction of finite element method used for the approximation of many differential and integral equations.

In the previous century, theory of existence, uniqueness and development of elementary methods based on power series were discussed. The study of partial differential equations also counted intensively. Bessel's, Legendre's, Hermite's, Chebyshev's and Hankel's functions among others came to know (Boyce and Diprima). By 1900 effective numerical methods devised and during last fifty years integration of technology enlarged the range of problems. In the meantime, numerical solver has been developed which acts as revolutionary aid to the solution of the differential equations. In the 20th century geometrical and topological method devised. The history of approximation methods for the solution of the boundary value problems is wide enough that it cannot be accumulated in this thesis work. However, some of the work is presented here.

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Although many numerical techniques available in literature for the solution of two point boundary value problems for ordinary differential equations, but in this work we will deal with some global methods and their comparison. Initially, we discuss the Pseudo-spectral Galerkin method and then a numerical problem is solved to find out the accuracy of said method. To compare our results, we have solved the same problem with shooting method, Finite difference method, Galerkin method, and Rayleigh-Ritz method (Stoer and Bulirsch, 2013; Smith, 1985; Donea, 1984; Bhat, 1985). Graphical results show the accuracy and durability of Pseudo-spectral Galerkin method. For different values of solution, error tables have been provided to see a clear comparison of method.

2. PSEUDO SPECTRAL GALERKIN METHOD

This method chooses the orthogonal polynomial as basis functions and takes the collocating points are the zeros of these polynomials.

Consider the model problem

$$u'' = f, \quad a \leq x \leq b \tag{1}$$

$$u(\pm 1) = 0 \tag{2}$$

Let

$$V = \{v(x) \in P_N \text{ and } v(\pm 1) = 0\} \tag{3}$$

be the space of Legendre interpolating polynomials over the real. We construct the finite dimensional subspace $V_N \subset V$ defined by

$$V_N = \{\phi_j, j = 0,1,\dots,N\}$$

V_N is a space of Nth degree Legendre interpolating polynomials with respect to the nodel values $\{x_j\}_0^N$ that is

$$-1 = x_0, x_1, x_2, \dots, \dots, x_{N-1}, x_N = 1$$

where $\{\xi_j\}_1^{N-1}$ with $x_0 = -1$ and $x_N = 1$ are the zeros of Legendre polynomial $P'_N(x)$. The $\{\eta_j\}_1^N$ are the nodel values of u_N that is

$$\eta_j = u_N(x_j), j = 1,2,\dots,N \tag{4}$$

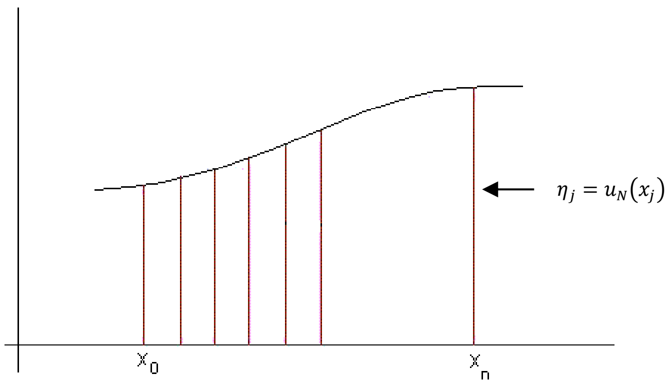

Figure 1: The basis functions ϕ_j are defined as

$$\phi_0(x) = \frac{(-1)^N(x-1)}{N(N+1)} P_N'(x)$$

$$\phi_j(x) = \frac{(x^2-1)}{N(N+1)P_N(x_j)(x-x_j)} P_N'(x), \quad 1 \leq j \leq N-1$$

$$\phi_N(x) = \frac{(x+1)}{N(N+1)} P_N'(x)$$

The pseudo spectral Galerkin's equation for the model problem is

$$\sum_{j=1}^{N-1} \left(\phi'_i, \phi'_j\right) \eta_j = (f, \phi_i) = \omega_i f(x_i) \qquad 1 \leq i \leq N-1 \tag{5}$$

are in the matrix form

$$A\eta = \underline{b}$$

The matrix A in the Pseudo-Spectral Galerkin's equation is called stiffness matrix and the vector b is called load vector.

3. METHODS COMPARISON

The comparison of the numerical techniques for the better choice is developed here. For this purpose, we have chosen the following example

$$-\frac{d^2y}{dx^2} + y(x) = -\frac{1}{20000} + \frac{1}{24000000}x, \tag{6}$$

Subject to the following boundary conditions

$$y(-1) = y(1) = 0 \tag{7}$$

The exact solution of the problem is given as

$$y(x) = \frac{1}{24000000(e^4-1)} \begin{bmatrix} 121e^{-x+3} - 119e^{-x+1} - 121e^{x+1} + \\ 119e^{x+3} + (e^4-1)x - 120(e^4-1) \end{bmatrix} \tag{8}$$

The error encountering in the approximation to the problem by using linear shooting method, finite difference method, Galerkin method, Raleigh-Ritz method and Pseudo-spectral Galerkin method for n=5 is given in the following table.

Table 1: Comparison table for N=5

Shooting Error	FD Error	Galerkin Error	Raleigh Ritz Error	PSG Glaerkin (Legendre) Error
1.4×10^{-6}	1.4×10^{-7}	2.6×10^{-9}	1.7×10^{-9}	2.7×10^{-12}
2.4×10^{-6}	3.0×10^{-7}	8.0×10^{-10}	1.9×9	1.3×10^{-12}
2.4×10^{-6}	5.2×10^{-7}	1.3×10^{-9}	1.9×10^{-9}	1.6×10^{-12}
1.4×10^{-6}	8.4×10^{-7}	1.4×10^{-8}	1.7×10^{-9}	2.9×10^{-12}

The above table gives value at the interior mesh point or collocating point. We can clearly see the spectral method is far better than the initial value method as well as finite difference method. Furthermore, graphical overview of the approximate solution to the above problem in comparison with exact solution is as follow.

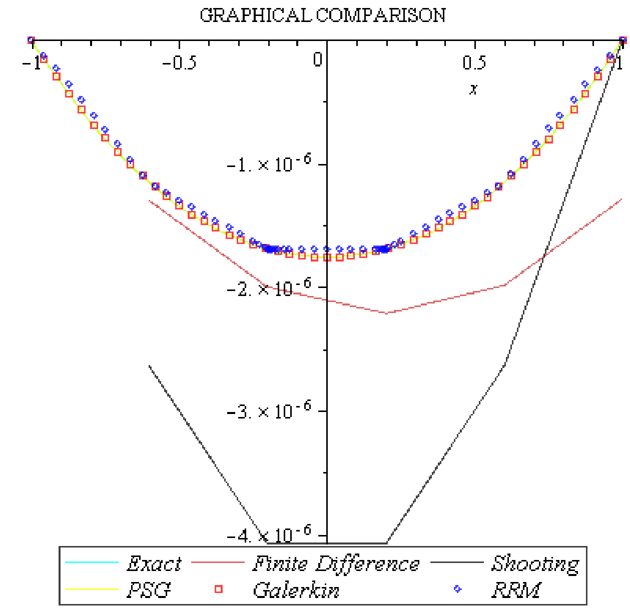

Figure 2: Graphical comparison at N=5

Although the error encountering by linear shooting and finite difference method is quite remarkable but in the comparative study with finite element method, we have come to know that for N=5 these methods are not trustworthy. Also, we will check the efficiency of these methods for greater values of N.

Table 2: Comparison table for N=7

Shooting Error	FD Error	Galerkin Error	Raleigh Ritz Error	PSG Glaerkin (Legendre) Error
1.1×10^{-6}	7.3×10^{-8}	3.1×10^{-9}	2.9×10^{-9}	1.9×10^{-13}
1.9×10^{-6}	1.5×10^{-7}	1.7×10^{-9}	5.0×10^{-9}	1.9×10^{-13}
2.4×10^{-6}	2.5×10^{-7}	5.7×10^{-10}	6.1×10^{-9}	2.2×10^{-13}
2.4×10^{-6}	3.7×10^{-7}	5.7×10^{-10}	6.1×10^{-9}	1.7×10^{-13}
1.9×10^{-6}	5.2×10^{-7}	1.5×10^{-9}	1.7×10^{-9}	1.1×10^{-13}
1.1×10^{-6}	7.1×10^{-7}	1.2×10^{-9}	2.9×10^{-9}	5.3×10^{-14}

For N=7 improvement can be in the approximation to the problem. Not only finite difference approximation improved also spectral methods marches towards accuracy.

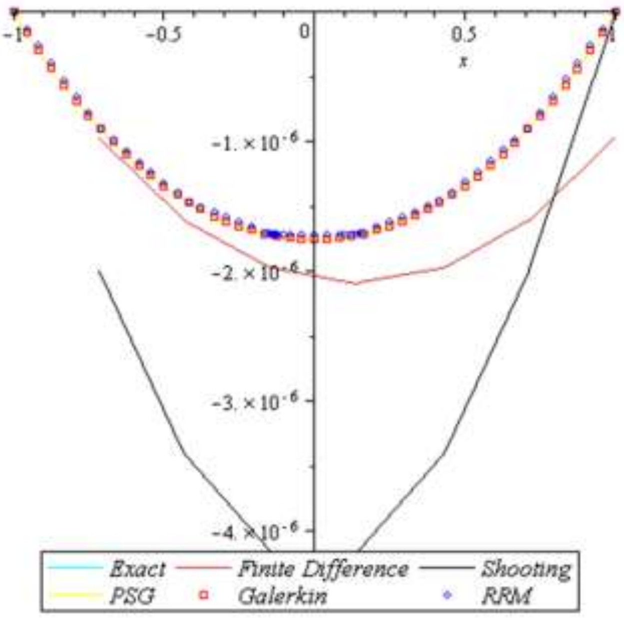

Figure 3: Graphical comparison at N=7

The following error table consists of value for N=9.

Table 3: Comparison table for N=9

Shooting Error	FD Error	Galerkin Error	Raleigh Ritz Error	PSG Glaerkin (Legendre) Error
8.6×10^{-7}	4.6×10^{-8}	3.5×10^{-9}	2.3×10^{-9}	8.8×10^{-13}
1.6×10^{-7}	9.4×10^{-8}	2.4×10^{-9}	3.7×10^{-9}	1.0×10^{-12}
2.2×10^{-6}	1.5×10^{-7}	1.4×10^{-9}	4.6×10^{-9}	1.2×10^{-12}
2.5×10^{-6}	2.1×10^{-7}	4.5×10^{-10}	5.1×10^{-9}	7.9×10^{-13}
2.5×10^{-6}	2.8×10^{-7}	4.6×10^{-10}	5.1×10^{-9}	4.5×10^{-13}
2.2×10^{-6}	3.7×10^{-7}	1.9×10^{-9}	4.6×10^{-9}	4.1×10^{-13}
1.6×10^{-6}	4.7×10^{-7}	8.1×10^{-9}	3.7×10^{-9}	1.3×10^{-13}
8.6×10^{-7}	6.1×10^{-7}	3.9×10^{-8}	2.2×10^{-9}	6.1×10^{-14}

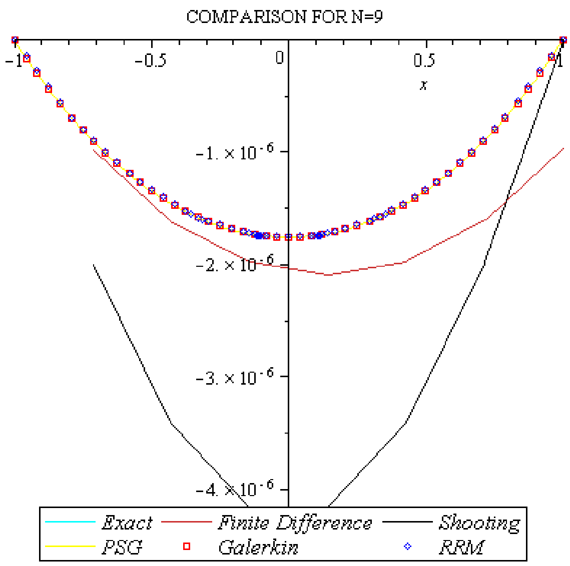

Figure 4: Graphical comparison at N=9

4. CONCLUSION

In this work, we have analyzed the Galerkin pseudo-spectral method by solving a numerical problem. In comparison with our results, the shooting method, the finite differential method, the Galerkin method and the Rayleigh-Ritz method are used to solve the same problem. The graphical results show the precision and reliability of the Galerkin Pseudo-spectral process. In error estimation, it can be seen that although the error encountering by linear shooting and finite difference method is quite remarkable but in the comparative study with finite element method, we have come to know that for lesser values of N, these methods are not trustworthy. As the value of N increases, the error of other methods becomes lower, but Galerkin pseudo-spectral method provides a high accuracy and rapid convergence even for small values of N.

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